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Bayesian graduation of mortality rates: An application to reserve evaluation

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Abstract

This paper presents Bayesian graduation models of mortality rates, using Markov chain Monte Carlo (MCMC) techniques. Graduated annual death probabilities are estimated through the predictive distribution of the number of deaths, which is assumed to follow a Poisson process, considering that all individuals in the same age class die independently and with the same probability. The resulting mortality tables are formulated through dynamic Bayesian models. Calculation of adequate reserve levels is exemplified, via MCMC, making use of the value at risk concept, demonstrating the importance of using "true" observed mortality figures for the population exposed to risk in determining the survival coverage rate.
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1. Introduction

Complementary open pension and life insurance plans that have survival coverage (retirement benefits) require prior approval for commercialization in Brazil. Open pension entities and life insurance companies must set the mortality tables and interest rates that will be applied to their plans at the time of obtaining this approval. However, the mortality tables must be valid from the adhesion of the first participant to the plan until the death of the last one, a very long interval. In this paper, we refer generically to the plans as pension plans and to the companies that sell them as insurers.

Among the problems faced by insurers, one of the chief ones is deciding which mortality table to establish for survival coverage of their plans, since this must be used to calculate contributions, benefits and reserves. This problem is aggravated by the trend toward greater longevity in the population at large and the use in Brazil of mortality tables constructed from the experiences of other countries, particularly the United States.

Insurers constitute mathematical reserves for future liabilities based on the values of the life annuities, as we shall see in Section 4 and the Appendix. These are calculated as a function of the interest rate and the probability of death. Hence, to avoid constituting mathematical reserves that are insufficient to cover their future commitments,

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and consequent insolvency, due to the lack of knowledge about the true mortality of the exposed population, the companies' actuaries have been putting aside statistical and actuarial considerations and trying to prevent future losses by setting reduced real interest rates, generally equal to zero. With this artifice, they hope to compensate for future technical losses arising from poorly formulated mortality tables, i.e., by using higher death probabilities than are really the case for the population covered by survival benefits, with financial gains from the guarantee of a low real interest rate. In this form, the Brazilian insurance market is becoming less technically grounded.

Based on data collected from all Brazilian insurers for the four years 1998–2001, a Bayesian statistical model is used to prepare mortality tables for both sexes that reflect the mortality rates of those exposed to risk who have survival coverage under pension plans in Brazil. To predict the probabilities of death contained in these tables, a Bayesian graduation process is used. The observed numbers of deaths are modelled as Poisson-distributed, assuming that all individuals of the same age die independently and with the same probability. Graduation is fundamental to smooth out the raw mortality rates so that the death probabilities are monotonically increasing with respect to age, since it is well known that human mortality behaves this way from a certain age onward.

The main objective of this paper is to propose hierarchical dynamic models for formulating Bayesian mortality tables (Gamerman and Migon, 1993). These models strongly differ from others found in literature from 1990 to 2000 since they are full Bayesian, dynamic and predictive in nature. They combine graduation with a dynamic description for the mortality evolution throughout time allowing to incorporate improvements in mortality rate. Certainly, they can be viewed as an extension of the graduation models introduced by Broffit (1988) and Carlin (1992). They also extend Renshaw and Haberman (2003) and Czado et al. (2005) allowing the parameters to vary throughout time and building mortality tables based on the predictive distribution of the number of deaths. The importance of better fitting the mortality tables used in calculating survival coverage to the reality of the exposed population, in order to maintain the solvency of insurers, is shown. This is done by calculating the mathematical reserve, comparing the "deterministic method" adopted by the Brazilian insurance market with our proposed Bayesian method, via stochastic simulation (MCMC), employing the value at risk (VaR) concept.

This paper is organized as follows. The data and concepts are presented in Section 2. In Section 3 dynamic models for preparing mortality tables are proposed. Modern techniques in model selection are also discussed in this section. In Section 4 the mathematical reserve is calculated by means of MCMC and value at risk concepts to demonstrate the importance of adjusting mortality tables to the real mortality of the exposed population so that the insurer can remain solvent. Finally, in Section 5 some conclusions and suggestions for extensions of this work are presented.

2. Graduation of mortality rates

The data used refer to participants in pension plans in Brazil with survival coverage for the years 1998–2001. We consider these data with the following structure: $D_t = (x, e_{x,t}, d_{x,t})$, where x is the age of the individuals in years, $e_{x,t}$ is the central number of those exposed to risk observed at age x in year t, and $d_{x,t}$ is the number of deaths observed at age x in year t, for $t = 1, \ldots, 4$, already corrected for any notification delays.

In the period under study, there were nearly 5.6 million men at risk and 2.9 million women. Of these 94% were between 20 and 60 years of age. In the same period, nearly 7600 men died and 2800 women, with the great majority, approximately 97%, doing so between the ages of 25 and 90. These data are available from the corresponding author on request. One would expect the mortality rate to behave regularly, with older individuals having higher probabilities of dying than younger people. In practice, however, the crude rates do not behave this way in relation to age, but form a noisy, irregular sequence. To be used in pension plans, the rates must be smoothed out. This process of smoothing out crude mortality rates is called graduation.

Graduation, besides correcting the problem mentioned above, serves to overcome the lack of information for some ages studied. The crude rates are smoothed so that the annual probabilities of death (q_x) are monotonically increasing with advancing age. This is a well accepted fact for human mortality rates, for ages x_l (the lower age value, for example $x_l = 20$ or 30) and above.

The crude mortality rates were graduated as a function of the force of mortality, i.e., the instantaneous variation in the intensity of death. Assuming that all individuals with the same age die independently and with the same probability, then the number of deaths observed, $d_{x,t}$, for each year and age, is Poisson-distributed with mean

$$E[d_{x,t} | \mu_{x,t}] = e_{x,t} \ \mu_{x,t}$$

where $\mu_{x,t}$ is the force of mortality at age x at time t, and $e_{x,t}$ denotes the exposed population, which is assumed known.

Since the aim of the graduation process is to obtain the annual probabilities of deaths for each age class and we are graduating as a function of the force of mortality, we consider these to be constant over constant age intervals and work with the following relation (Gerber, 1997):

$$q_x = 1 - \exp(-\mu_x). \tag{1}$$

To estimate the smoothed mortality rates, the literature suggests parametric and non-parametric graduation, in the nomenclature of Haberman and Renshaw (1996), or global and local models, respectively. The local models are related to a semi-parametric spline smoothing (Kohn and Ansley, 1987). Graduations are performed, in this paper, through parametric models, which we call global models, by adjusting the probabilities of death or of mortality forces to a mathematical model. The most common mathematical models represent survival functions based on laws of mortality, such as those of de Moivre, Gompertz, Makeham and Weibull (Bowers et al., 1986). A broad class of Bayesian generalized growth models, which accommodates the mortality law mentioned before, is described in Migon and Gamerman (1993).

In this paper, however, we propose dynamic Bayesian models, with two prior modelling alternatives: local and global, to take care of the smoothness of the crude rates. The Bayesian approach to the graduation process involves statistical estimation of the unknown parameters, where initial knowledge of the parameters of interest (prior distribution) is aggregated to the data. Some works have already appeared on Bayesian graduation, among them Kimeldorf and Jones (1967), Broffit (1988), Carlin (1992) and Mendoza et al. (2001).

3. Proposed models

A mortality table is defined as a set of annual probabilities of death graduated by age. Its construction must consider a period of study greater than one year. Haberman and Renshaw (1996) and Renshaw and Haberman (2003) use four years, and some tables available in the market use up to six years of data. Our models can also consider any number of years of data but give more strength to the recent ones, recognizing that present data are more informative than the older ones. A broad class of dynamic models for preparing Bayesian mortality tables to apply to the exposed population having survival coverage in Brazil is proposed. We extend the model presented by Broffit (1988) to include the temporal dimension. Hence, to describe the relation between the number of observed deaths and the corresponding ages, we use the following model:

$$d_{x,t} \sim \text{Po}(e_{x,t}\mu_{x,t}), \quad x = x_l, \ldots, x_u \tag{2}$$

where x_l and x_u denote, respectively, the lower and the upper age limits considered, $t = 1, \ldots, T$, the number or periods of observation, $\mu_{x,t} > 0$ and $e_{x,t}$ are known constants and $\text{Po}(\lambda)$ denotes a Poisson distribution with mean $\lambda > 0$. Two alternatives are used in the modelling: local and global.

3.1. Local dynamic model

To model the evolution over time of the mortality forces, we use generalized dynamic models (West et al., 1985), considering $\mu_{x,t}$ related through multiplicative perturbations:

$$\log(\mu_{x,t}) = \log(\mu_{x,t-1}) + \omega_t, \quad \omega_t \sim N[0, W_t]$$

where $x = x_l, \ldots, x_u,\ t = 2, \ldots, T$, W_t is modelled through non-informative Inverse Gamma distribution. We stress that if $W_t = 0$, we revert to a static model. The above equation explicitly considers that death probability evolves in time smoothly. In dynamic modelling and Bayesian forecasting nomenclature, this is called a first order or steady state model. In this case, the data over the years are considered with the same relevance, as if it were a regression with replications in each age bracket. The model is completed by specifying vague prior Gamma distributions for the parameters $\mu_{x,1}$. So, we have:

$$\mu_{x,1} \sim Ga(\alpha, \beta) I_{(\mu_{x-1,1}, \mu_{x,1})}(\mu_{x,1}), \quad x = x_l, \ldots, x_u$$

where $I_A(y)$ is the indicator function, assuming the values 1, if $y \in A$ and zero otherwise. The hyperparameters $\alpha, \beta > 0$ are assumed known, unlike in Carlin (1992), only because of the computational limitations of present version of WinBUGS (Spiegelhalter et al., 2003). Nevertheless, the posterior estimates of the mortality rates are almost insensitive to variations on the hyperparameters α and β.

It should be observed that, by graduating $\mu_{x,1}$, all $\mu_{x,t}, \forall t \geq 2$ will also be graduated, due to the evolution of the parameters (2). Besides this, Bayes' Theorem assures that the restrictions on the prior will also be valid for the posterior (Carlin, 1992), in this way producing graduated Bayesian estimators for all age classes through time. A more recent application of these concepts, in the context of estimating demographic models with restrictions in the parameters, can be found in McDonald and Prevost (1997).

3.2. Global dynamic model

The graduation in this case is done by adjusting the mortality rates to Makeham's Law (Bowers et al., 1986), permitting the parameters to evolve smoothly over time:

$$\mu_{x,t} = \alpha_t + \beta_t \delta_t^x, \tag{3}$$

with $x = x_l, \ldots, x_u$ and $t = 1, \ldots, T$.

Starting from the canonic link function $(\eta_{x,t})$ of the Poisson distribution, as in generalized dynamic linear models, we obtain the structure:

$$\eta_{x,t} = \log(e_{x,t}) + \log(\alpha_t + \beta_t \delta_t^x),$$

with $x = x_l, \ldots, x_u$ and $t = 1, \ldots, T$, where $\mu_{x,t} > 0$ and $e_{x,t}$ are known constants.

The time evolution of the parameters that constitute Makeham's Law are described by

$$\begin{aligned}
\log(\alpha_t) &= \log(\alpha_{t-1}) + \omega_{\alpha,t} \\
\log(\beta_t) &= \log(\beta_{t-1}) + \omega_{\beta,t} \\
\log(\delta_t) &= \log(\delta_{t-1}) + \omega_{\delta,t}, \quad t = 2, \ldots, T
\end{aligned}$$

with $\omega_{\theta,t} \sim N[0, W_{\theta,t}]$, where θ generically denotes the state parameter and $W_{\theta,t}$ is modelled through non-informative Inverse Gamma distribution. Once again, static models are a particular case corresponding to $W_{\alpha,t} = 0$, $W_{\beta,t} = 0$ and $W_{\delta,t} = 0$. The model is completed by attributing non-informative prior normal distributions to the parameters: α_t, β_t and δ_t, $t = 1$, satisfying the restrictions imposed by Makeham's Law. Allowing the parameters α, β and δ to be time varying ensures that the more recent data are considered in the estimate process with large weights. Then, any improvement in mortality rates will be taken into account.

3.3. Predictive distribution of the number of deaths

The predictive distribution of the number of deaths at time $T + 1$, for each age class, $x = x_l, \ldots, x_u$, is given by

$$p(d_{x,T+1}|D_T) = \int p(d_{x,T+1}|\mu_{x,T+1}) p(\mu_{x,T+1}|D_T) \mathrm{d}\mu_{x,T+1} \tag{4}$$

since $d_{x,T+1} \perp D_T | \mu_{x,T+1}$, where $\mu_{x,t} > 0$, T is the number of observation periods and represents the total available information. In this way all the uncertainties associated with the non-observable parameters are incorporated in this calculation.

The predictive distributions for the probabilities of death of each age class, involved in the calculation of the k-year pure endowment (see the Appendix), are obtained based on the concept of predictive distribution described in Eq. (4), and applying the simplification described in Eq. (1), assuming that $e_{x,t}$ are known constants:

$$q_{x,T+1} = 1 - \exp(\mu_{x,T+1}), \quad x = x_l, \ldots, x_u. \tag{5}$$

Hence, the means of the predicted sample values for this quantity are considered the Bayesian estimators of the probabilities of future deaths.

Table 1
Results of the comparison methods of the proposed models, for both sexes (10^3)

Dynamic models	EPD			DIC			LS
	$G(m)$	$P(m)$	$D(m)$	$\bar{D}$	pD	DIC	LS
Male							
Local	30.50	8.12	38.62	2.17	0.024	2.19	−1.13
Global	20.94	7.67	28.61	1.78	0.007	1.79	−0.91
Female							
Local	9.38	2.96	12.34	1.65	0.019	1.67	−0.87
Global	7.09	2.82	9.91	1.41	0.008	1.42	−0.73

3.4. Implementation and analysis of convergence

The models introduced before do not allow for full inference in so far the posterior and predictive distributions of the unknown quantities of interest are not available in closed form. Often analytical unsolved problems in Bayesian inference can be approximately solved by sampling from the relevant posterior distribution. The distributions involved here are too complex for directly drawing samples from. A stochastic simulation approach, named Markov chain Monte Carlo (MCMC), is used in this application. A complete consideration of MCMC at an expository level can be found in Gamerman (1997). This is a collection of sampling techniques that have revolutionized the Bayesian statistics in the last decades. Roughly it deals with sampling from a complex distribution in two steps: (i) firstly, a Markov chain with a transition kernel such that the limiting distribution is given by the joint posterior distribution is constructed; (ii) in the next step, a trajectory is sampled from this chain. For a suitably large iteration, values are virtually sampled from the posterior distribution. The two most used transition kernels are Gibbs sampling and component-wise Metropolis–Hastings steps. In the first case the kernel is composed of product of the posterior full conditionals of parameters divided in blocks, while in the second case, the kernel is still composed of products of proposal full conditionals, not necessarily derived from the posterior. It is also a common practice to implement many parallel chains, starting at different points of the parameter space hoping they mix.

We implemented the proposed models through MCMC, particularly Gibbs sampling, using *WinBUGS* version 1.4 (Spiegelhalter et al., 2003). Due to the characteristics of the data, we estimated the probabilities of death for ages between 25 and 90 years ($x_l = 25$ and $x_u = 90$), considering an observable horizon of four years ($T = 4$), between 1998 and 2001. We should mention that there were no recorded deaths for some of these ages. These are dealt with easily in the inference process used in this paper. In implementing Gibbs sampling, we used three parallel chains with different initial values, seeking to keep the samples generated from clustering in regions around a local mode, in the case of posterior multimodality, as described in Gamerman (1997). Verification of convergence is an important step in applying these methods. Here we use three informal graphical analysis techniques (density, autocorrelation function and traces) and the Gelman–Rubin statistic, as modified by Brooks and Gelman (1998), and as applied by Migon and Moura (2005). For each parameter we generated a sample of 120,000 observations, through three parallel chains, of which the first 60,000 were discarded. It took around 15 min to implement the local dynamic model, on a PC with 1.8 GHz and 512 MB of RAM. The global dynamic model took roughly three times as long.

3.5. Selection of models and analysis of results

Selection of models is fundamental in modern statistics (Gelfand and Ghosh, 1998) due to computational advances that allow fine-tuning multiple complex and alternative models. We apply three model selection methods to our data to identify the best model for preparing mortality tables. Besides the Bayes factor, calculated from the predictive log likelihood (LS) of each model estimated by the harmonic mean of the likelihood values (Newton and Raftery, 1994), we use two other methods. These criteria simultaneously consider the fitting of the observed data and the prediction of replicated data, and are EPD (expected predictive deviance), proposed by Gelfand and Ghosh (1998), and DIC (deviance information criterion), developed by Spiegelhalter et al. (2002). Table 1 shows the results obtained by the methods of comparison of the proposed models.

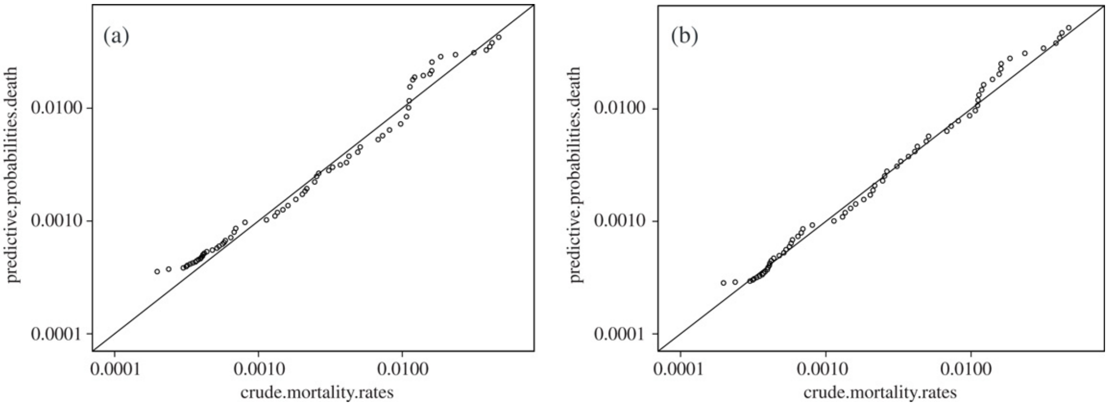

Fig. 1. Quantile–quantile plots for observed versus estimated probabilities of death (logarithmic scale) for males based on: (a) local dynamic model and (b) global dynamic model.

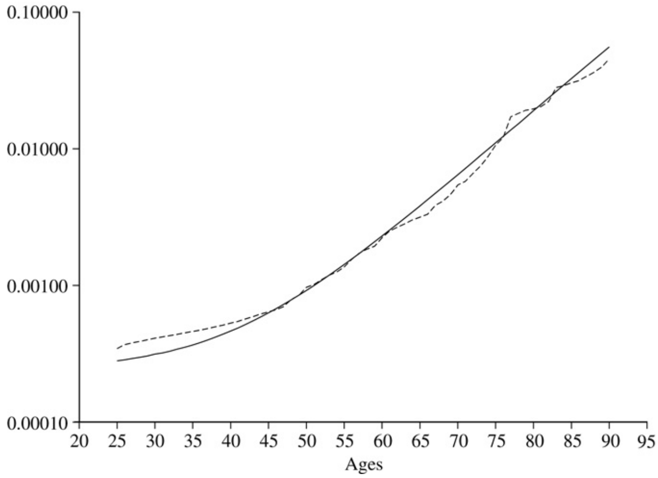

Fig. 2. Estimated probabilities of death (logarithmic scale) based on the proposed models for males, from the global dynamic model (solid line) and local dynamic model (dotted line).

In Fig. 1 we present the quantile–quantile plots for observed versus fitted death probabilities obtained using the local and global models, for males. The global model is slightly better than the local one.

We conclude that regardless of gender and the comparison method employed, the global model is the best for preparing Bayesian mortality tables. The estimated probabilities of death, as well as their 95% probability intervals obtained by this model, for males and females, can be obtained on request. Besides this, we can see that the estimated probabilities of death by this model are smoother than those obtained with the local model (Fig. 2).

4. Effect of Bayesian mortality tables in the reserve evaluation

In this section we show the importance of adjusting the mortality table used in survival coverage to the real mortality rates of the exposed population in order for insurers to remain solvent. We do this by calculating the mathematical reserve by means of MCMC, using value at risk (VaR) concepts. These reserves reflect the future commitment of the insurer less that of participants, and is divided into the mathematical reserve for benefits to be granted (RBG), constituted before granting the benefit, and the mathematical reserve for benefits already granted (RBAG), constituted while the beneficiary is receiving benefit. The RBAG reflects only the insurer's total future obligations, since the beneficiaries are no longer contributing to the plan, and is calculated by multiplying the value of the benefit by the value of the life annuity. In Brazil, payments are usually made monthly and the RBAG is calculated and accounted for monthly. Hence, the monthly calculation of this value consists of multiplying the value of the monthly benefit by the value of the monthly life annuity.

A life annuity is defined as the present actuarial value of a series of unit-value payments. There are various types of life annuities, which can be classified regarding frequency of payment, form of payment, payment schedule (at the start or end of each period).

Regardless of their classification, life annuities are calculated based on the mortality tables, interest rate and methodology established in the pension plan, subject to approval by the Brazilian regulatory agency.

We now compare the mathematical reserve for benefits already granted, considering benefit is to be paid for life at the start of each month, obtained by means of the "deterministic method" used by the insurance market, considering the mortality table established in the plan; and also by calculating the predictive distribution of the life annuity obtained by stochastic simulation. We use the Bayesian mortality table chosen in the previous section and the concept of value at risk (VaR) to determine the amount necessary to guarantee the insurer's solvency.

The "deterministic" calculation of the monthly whole life annuity-due to a life aged x, usually denoted $12\ddot{a}_x^{12}$, is summarized in the Appendix. It represents the present actuarial value of the unit-value payments made for life at the start of each month to an individual of age x.

Our goal, in this application, is to use the MCMC outputs to obtain the predictive distribution of the life annuity-due, denoted by $LA_x = 12\ddot{a}_x^{12}$ (see Eq. (6) in the Appendix). This is a function of the interest rate, i, (assumed known and fixed) and of the probabilities of death from age x up to age x_u, $(q_x, \ldots, q_{x_u})_{T+1}$. These probability are obtained from the MCMC outputs. Denote the full available information by $D_T^* = D_T \cup \{e_{T+1}\}$, the state parameters of the graduation model by θ and use, at time $T+1$, the following steps:

(i) given some initial values for $(\theta^{(0)}, d^{(0)})$, for each age class x, draw:
   (a) the θ's from the full conditional posterior,

$$\theta^{(j)} \sim p(\theta|\theta^{(j-1)}, D_T^*),$$

   (b) the future number of deaths $d^{(j)}$ from

$$d^{(j)} \sim p(d|\theta^{(j-1)}, D_T^*), \quad \forall j = 1, \ldots, N.$$

(ii) calculate $\mu^{(j)}(\theta^{(j)})$ using Eq. (4) and $q_x^{(j)}$ via Eq. (5).
(iii) obtain $LA_x^{(j)}$ as a function of the $(q_x^{(j)}, \ldots, q_{x_u}^{(j)}, i)$.

These steps complete the MCMC algorithmic cycle and determine the desired predictive distribution of the life annuity-due, $p(LA_x \mid D_T)$. The value at risk (VaR), percentile of order $1-\alpha$ (α is the level of risk) of the above predictive distribution will be denoted by $LA_{x,\alpha}$ and is easily obtained from the MCMC outputs. It corresponds to a point estimated of the life annuity for age x. The level of risks to be used must be chosen in accordance with company management or the decision maker. So, the RBAG value necessary for the insurer to remain solvent is calculated by multiplying the VaR by the monthly lifetime benefit. The difference between this and the RBAG calculated by means of the mortality table, using the "deterministic" methodology approved in the plan, must be borne by the insurer. This amount corresponds to the additional reserve for solvency, which is called reserve for insufficient contribution (RIC). Constitution of such a reserve is mandatory by force of specific regulation. Below we exemplify the importance of adjusting the mortality table by calculating the reserve, making use of value at risk concepts.

4.1. Example of calculating reserve for a single participant

Given a hypothetical pension plan that guarantees 6% real interest per year and uses the AT-83 Male table, let us calculate the reserves for a $x = 60$-year old male beneficiary who receives a monthly lifetime benefit at the start of each month of R$1000.00. Applying Eq. (6) in the Appendix, we obtain a monthly whole life annuity-due of 141.34. The mathematical reserve for benefits already granted is R$141,340.00. Using the outputs of the MCMC implemented in *WinBUGS*, we obtain the predictive distribution shown in Fig. 3.

Setting the level of risk, we can obtain the value of the RBAG necessary for the insurer to remain solvent and the reserve to be constituted (RIC). Table 2 shows these quantities for different values of the level of risk. We can also

Table 2
Calculation of life annuities and reserves, in Brazilian currency (reais), using the VaR concept, for different levels of risk

Level of risk (%)	$LA_{x,\alpha}$	RBAG	RIC
50	159.6	159,600.00	18,260.00
25	160.0	160,000.00	18,660.00
10	160.3	160,300.00	18,960.00
5.0	160.5	160,500.00	19,160.00
2.5	160.6	160,600.00	19,260.00
1.0	160.8	160,800.00	19,460.00

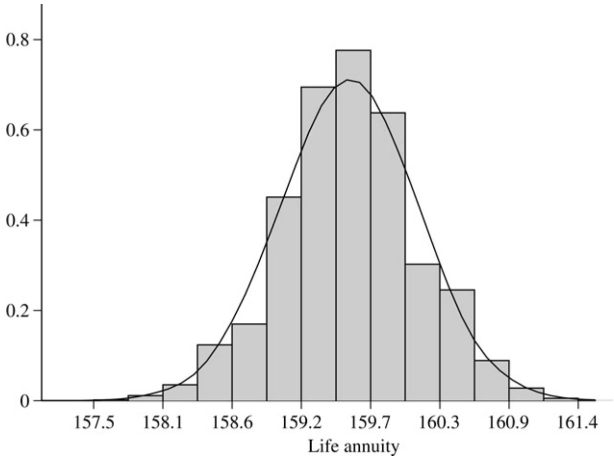

Fig. 3. Predictive density distribution of the monthly whole life annuity-due to a male, age 60.

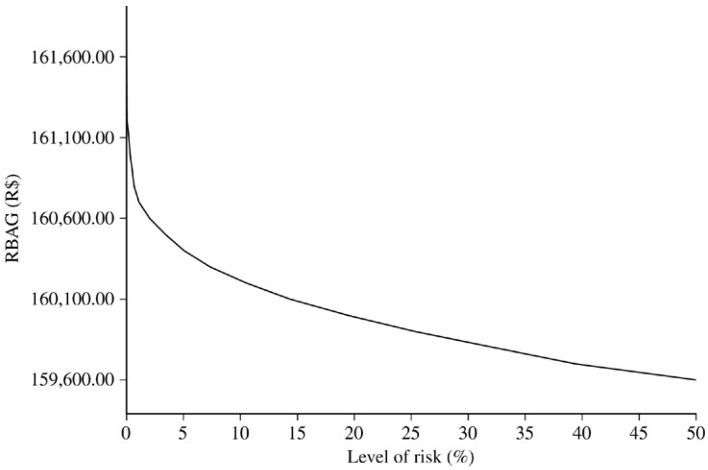

Fig. 4. RBAG necessary for an insurer to remain solvent, in Brazilian currency (reais), as a function of the level of risk.

verify the evolution of the RBAG calculated by the Bayesian method as a function of the increase in the level of risk by analyzing Fig. 4.

Analyzing Table 2 and Fig. 4, we can see that the lower the risk the insurer wants to incur, the greater the RBAG necessary for it to remain solvent and the greater the RIC it must fund. Hence, the task of choosing the level of risk is very important to manage the company with an adequate safety margin.

5. Conclusion

In this paper we implement Bayesian graduation models of mortality rates using MCMC techniques, calculated with *WinBUGS*. The probabilities of future deaths are estimated by means of the predictive distribution of the number of deaths for each age, which is modelled as being Poisson-distributed, considering that all individuals of the same age die independently and with the same probability. To construct Bayesian mortality tables, we propose two dynamic models – local and global – considering the temporal evolution of mortality occurring in the sample population in the years 1998–2001 to estimate the death probabilities for 2002. After comparing the models, we find that the dynamic global model is best for preparing Bayesian mortality tables for both sexes.

With the aim of exemplifying the importance of adjusting the mortality table used in survival coverage to the real mortality of the exposed population in order to avoid future insolvency, we compare the calculations of the mathematical reserves for benefits already granted, obtained by the following methods:

1. "Deterministic" (employed by the insurance market), using a known mortality table; and
2. Bayesian, using MCMC techniques and VaR concepts.

From this comparison we calculate the additional reserve (RIC) insurers should constitute to remain solvent. We hope with this to help insurers evaluate the true mortality rates of their exposed populations, by proposing graduated Bayesian models implemented through a statistical package (*WinBUGS*) that can be obtained at no cost. This will enable them to adopt mortality tables that are better adjusted to reality and hence to adequately constitute their reserves to avoid future insolvency. A natural extension of this work, in view of the tendency for people to live longer, would be to model the mortality reduction factor, as done by Renshaw and Haberman (2003), but with a Bayesian and predictive focus, unlike the model described by these authors. However, since we have only four years of data for the Brazilian market, at the moment this modeling is impossible.

Appendix A

Let us succinctly describe the deterministic calculation of the monthly whole life annuity-due to a life aged x ($12\ddot{a}_x^{12}$), which represents the present actuarial value of the unit-value payments made for life at the start of each month to an individual of age x. We let ${}_kE_x$ denote the value of the actuarial discount, i.e., the present value of a unit value of benefit owed at time k to a life aged x. This is called k-year pure endowment:

$$ {}_kE_x = \frac{1}{(1+i)^k}\ {}_kp_x $$

where ${}_0E_x = 1$, i is the real interest rate per year and ${}_kp_x$ is the probability that an individual of age x will survive until age $x+k$.

Hence, the annual whole life annuity-due to a life aged x is given by

$$ \ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \frac{N_x}{D_x} $$

where $N_x = \sum_{k=0}^{\infty} D_{x+k}$, $D_x = l_x \frac{1}{(1+i)^x}$, and $l_x = l_{x-1}q_{x-1}$, with l_x the number of survivals to each age x and l_0 the root of the table, set, without loss of generality, at 10,000. On the other hand, the monthly whole life annuity-due to a life aged x will be (Bowers et al., 1986)

$$ 12\ddot{a}_x^{12} = \sum_{k=0}^{\infty} \frac{1}{(1+i)^{\frac{k}{12}}}\ {}_{\frac{k}{12}}p_x \simeq 12\left(\ddot{a}_x - \frac{11}{24}\right) \tag{6} $$

where $\ddot{a}_x^{12}$ is the present actuarial value of a life annuity-due of 1 payable twelve times a year to a life aged x.

Appendix B

See Table 3.

Table 3
Central number of exposed to risk (e_x) and number of deaths observed (d_x) from 1998 to 2001 (Male: M, Female: F)

x	M		F		x	M		F		x	M		F	
	e_x	d_x	e_x	d_x		e_x	d_x	e_x	d_x		e_x	d_x	e_x	d_x
0	1.457	0	1.149	0	39	191.767	131	93.594	33	78	6.513	172	3.814	56
1	2.002	0	1.594	0	40	189.534	138	91.644	48	79	5.239	165	3.182	50
2	2.290	0	1.877	0	41	189.988	131	92.189	31	80	4.132	101	2.586	45
3	2.207	0	1.843	0	42	187.871	151	90.231	40	81	3.165	75	2.142	53
4	2.149	0	1.852	0	43	181.762	142	87.304	47	82	2.233	59	1.489	24
5	2.112	0	1.788	0	44	179.907	158	85.047	42	83	1.602	91	1.073	33
6	2.023	0	1.789	0	45	169.827	147	80.524	30	84	1.231	41	797	20
7	1.950	0	1.736	0	46	165.711	143	77.097	42	85	972	45	603	17
8	1.853	0	1.668	2	47	158.260	137	72.922	59	86	728	19	465	18
9	1.883	0	1.628	0	48	148.063	165	67.730	76	87	644	32	388	11
10	1.902	0	1.733	0	49	138.978	151	62.511	59	88	517	19	288	5
11	1.945	0	1.716	0	50	127.568	183	57.626	52	89	433	19	206	6
12	1.814	0	1.682	0	51	118.329	149	54.355	55	90	330	15	153	5
13	1.781	0	1.653	0	52	107.069	161	49.756	49	91	205	12	118	6
14	2.362	1	2.074	0	53	98.261	162	45.502	40	92	139	11	90	5
15	4.397	0	3.735	3	54	86.882	140	41.577	48	93	86	8	72	2
16	7.537	3	6.117	6	55	78.143	140	37.983	56	94	55	4	46	0
17	11.602	3	9.251	2	56	69.787	148	34.244	34	95	41	4	33	1
18	17.668	10	13.668	10	57	61.254	148	29.809	56	96	23	0	24	1
19	27.282	15	20.373	5	58	54.390	140	26.609	70	97	14	0	17	0
20	39.187	22	26.421	7	59	49.762	117	23.674	47	98	16	0	27	0
21	52.114	21	32.970	14	60	44.583	136	21.699	53	99	10	0	18	2
22	63.735	34	38.559	11	61	41.545	146	20.216	38	100	8	1	5	0
23	71.537	42	42.467	19	62	36.775	132	18.005	39	101	3	1	3	0
24	79.096	48	46.490	9	63	33.728	125	16.471	42	102	0	1	1	0
25	86.551	53	50.129	13	64	30.998	134	15.066	56	103	0	0	0	0
26	94.143	66	54.948	19	65	27.926	116	13.777	54	104	0	0	0	0
27	104.969	64	60.760	12	66	24.595	93	12.696	46	105	0	0	0	0
28	114.155	57	65.441	18	67	21.735	119	11.593	42	106	0	0	0	0
29	125.502	64	72.160	29	68	19.791	105	11.251	60	107	0	0	0	0
30	132.145	83	74.291	21	69	18.174	109	10.590	50	108	0	0	0	0
31	142.684	84	78.735	25	70	15.824	126	9.683	47	109	0	0	0	0
32	149.030	90	79.380	21	71	13.606	93	8.616	58	110	0	0	0	0
33	160.011	88	83.987	24	72	11.522	102	7.608	66	111	0	0	0	0
34	169.838	115	88.296	29	73	10.142	100	6.739	65	112	0	0	0	0
35	179.808	111	91.186	44	74	9.311	109	6.104	70	113	0	0	0	0
36	188.322	115	94.700	28	75	8.779	129	5.504	72	114	0	0	0	0
37	191.326	122	95.452	31	76	8.015	130	4.871	64	115	1	0	0	0
38	191.510	130	94.608	45	77	7.233	187	4.213	57	116	1	0	0	0

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