Sample Problem Sheet

Nicola Talbot

December 10, 2011

1. Differentiate from first principles $f(x) = \sqrt{x}$

   Solution:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \delta x} + \sqrt{x})}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

2. Differentiate the following functions:

   (a) $y = \arcsin(x)$

   Solution:

$$\sin(y) = x$$

   diff. w.r.t. x:

$$\begin{aligned}
\cos y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1 - \sin^2 y}} \\
&= \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}$$

   (b) $y = \arctan x = \tan^{-1} x$

   Solution:

$$\tan y = x$$

diff w.r.t. x:

$$\begin{aligned}
\sec^2 y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}
\end{aligned}$$

(c) $f(x) = g(x)^{h(x)}$.
Solution:

$$\begin{aligned}
f(x) &= e^{\ln g(x)^{h(x)}} \\
&= e^{h(x)\ln g(x)} \\
f'(x) &= e^{h(x)\ln g(x)}(h'(x)\ln g(x) + h(x)\frac{g'(x)}{g(x)}) \\
&= g(x)^{h(x)}(h'(x)\ln g(x) + \frac{h(x)g'(x)}{g(x)})
\end{aligned}$$

(d) $y = (\tan x)^{-1} = \cot x$
Solution:

$$\begin{aligned}
\frac{dy}{dx} &= -(\tan x)^{-2}\sec^2 x \\
&= -\frac{\cos^2 x}{\sin^2 x}\cdot\frac{1}{\cos^2 x} \\
&= \frac{-1}{\sin^2 x} \\
&= -\csc^2 x.
\end{aligned}$$

(e) $y = \cos(x^2)\sin x$.
Solution:
$$\frac{dy}{dx} = -\sin(x^2)2x\sin x + \cos(x^2)\cos x$$

3. Differentiate w.r.t. x:
$$e^{xy} = 2x + y$$

Solution: Differentiating both sides w.r.t. x:

$$\begin{aligned}
e^{xy}(1y + x\frac{dy}{dx}) &= 2 + \frac{dy}{dx} \\
xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} &= 2 - ye^{xy} \\
\frac{dy}{dx}(xe^{xy} - 1) &= 2 - ye^{xy} \\
\frac{dy}{dx} &= \frac{2 - ye^{xy}}{xe^{xy} - 1}
\end{aligned}$$

4. Find the gradient of the unit circle ($x^2 + y^2 = 1$).

   Solution: Differentiating with respect to x gives:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= \frac{-x}{\sqrt{1-x^2}}. \end{aligned}$$

5. Under which of the following functions does $S = \{a_1, a_2\}$ become a probability space?

   (a) $P(a_1) = \frac{1}{3}$, $P(a_2) = \frac{1}{2}$ $\boxed{\text{(b)}}$ $P(a_1) = \frac{3}{4}$, $P(a_2) = \frac{1}{4}$

   $\boxed{\text{(c)}}$ $P(a_1) = 1$, $P(a_2) = 0$ (d) $P(a_1) = \frac{5}{4}$, $P(a_2) = -\frac{1}{4}$

6. A coin is weighted so that heads is four times as likely as tails. Find the probability that: (a) tails appears, (b) heads appears

   Solution: Let $p = P(T)$, then $P(H) = 4p$. We require $P(H) + P(T) = 1$, so $4p + p = 1$, hence $p = \frac{1}{5}$. Therefore: (a) $P(T) = \frac{1}{5}$, (b) $P(H) = \frac{4}{5}$

7. Which of the following is the derivative of $x \sin(x)$? (Circle the correct answer.)

   Solution:

   A $\sin(x)$

   B $x\cos(x)$

   C $\sin(x) + x\cos(x)$ (product rule).

8. Describe what is meant by object-oriented programming.